Thomas' calculus (12th edition) by George B. Thomas, Jr.

p. 791 Q60, 65

60. Let

$$f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0,0)$.

65.

Find the value of $\frac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation

$$xy + z^3 x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

(MATH 2010 D Problem set 3, Q4).

4. Let $S = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Show that

(a) $\text{Int}(S) = \phi$;

(b) $\partial S = S$;

(c) $\text{Ext}(S) = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \neq 0\}$

Proof of Clairaut's Theorem using mean value theorem.

60. Let

$$f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0,0)$

Solution:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{\frac{\sin(x^3)}{x^2} - 0}{x}$$

$$= \lim_{x \to 0} \frac{\sin x^3}{x^3} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{\frac{\sin(y^4)}{y^2} - 0}{y}$$

$$= \lim_{y \to 0} \frac{\sin y^4}{y^3} = 0$$

65. Find the value of $\frac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation

$$xy + z^3x - 2yz = 0 \quad \text{———} \quad ①$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution.

Regarding z as a function of two independent variables x, y, $\quad z = z(x,y)$

Differentiate both sides of ① with respect to (w.r.t) x.

$$y + \left(3z^2 \frac{\partial z}{\partial x} \cdot x + z^3\right) - 2y\frac{\partial z}{\partial x} = 0 \quad \text{———} \quad ②$$

Substitute $(x,y,z) = (1,1,1)$ in ② :

$$1 + \left(3\frac{\partial z}{\partial x} + 1\right) - 2\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(1,1,1) = -2$$

4. Let $S = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Show that

(a) $\text{Int}(S) = \phi$;

(b) $\partial S = S$;

(c) $\text{Ext}(S) = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \neq 0\} = \mathbb{R}^2 \setminus S$

Solution :

(a). Let $x \in \mathbb{R}$ (let some point $(x,0) \in S$)

Let $\varepsilon > 0$, $(x, \frac{\varepsilon}{2}) \in B_\varepsilon(x,0)$

but $(x, \frac{\varepsilon}{2}) \notin S$

$\therefore$ $B_\varepsilon(x,0) \not\subseteq S$ for any $\varepsilon > 0$.

For any $x \in \mathbb{R}$, $(x,0)$ is NOT an interior point of S. $\therefore$ $\text{Int}(S) = \phi$

(c) Let $(x_0, y_0) \in \mathbb{R}^2$ with $y_0 \neq 0$ (i.e. $(x_0,y_0) \notin S$)

Let $\varepsilon = \frac{|y_0|}{2} > 0$.

For any $(x,y) \in B_\varepsilon(x_0,y_0)$, we want to claim that $(x,y) \notin S$.

This will imply that $\text{Ext}(S) = \mathbb{R}^2 \setminus S$

To finish the proof,

$$|(x,y)-(x_0,y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$
$$< \varepsilon = \frac{|y_0|}{2}$$

$$|y-y_0| \leq \sqrt{(x-x_0)^2+(y-y_0)^2} < \frac{|y_0|}{2}$$

(Δ-inequality) $|y_0|-|y| \leq |y-y_0| < \frac{|y_0|}{2}$

$$|y| > \frac{|y_0|}{2} > 0$$

(b) Any points in $\mathbb{R}^2$ must be one of the following three types :

① Interior point $\text{Int}(S)$
② Exterior point $\text{Ext}(S)$
③ Boundary ∂S

$\therefore$ By (a) and (c), $\partial S = S$.

Proof of Clairaut's theorem :

Clairaut's theorem :

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Let $(a,b) \in \mathbb{R}^2$.

Suppose $\partial_x f$, $\partial_y f$, $\frac{\partial^2 f}{\partial x \partial y}$ exist everywhere.

If $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (a,b),

then $\frac{\partial^2 f}{\partial y \partial x}(a,b)$ exists and

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b)$$

Observation :

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \lim_{h \to 0} \frac{1}{h}\left[\frac{\partial f}{\partial y}(a+h,b) - \frac{\partial f}{\partial y}(a,b)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\lim_{k \to 0} \frac{1}{k}\Big(f(a+h,b+k) - f(a+h,b)\Big) - \frac{1}{k}\Big(f(a,b+k) - f(a,b)\Big)\right]$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{1}{hk}\left[f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)\right]$$

Similarly,

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) = \lim_{k\to 0} \lim_{h\to 0} \frac{1}{hk}\left[f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)\right]$$

Just the order of limit changes
(first taking $\lim_{h\to 0}$, then $\lim_{k\to 0}$ $\Rightarrow$ $\frac{\partial^2 f}{\partial y \partial x}$)

Now, we let

$$\Delta(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$$

Apply Mean value theorem

$$\Delta(h,k) - \Delta(h,0) = k\,\partial_k \Delta(h, tk) \quad \text{for some } t \in (0,1)$$

$$= k\left[\partial_y f(a+h, b+tk) - 0 - \partial_y f(a, b+tk) + 0\right]$$

Applied chain rule for partial derivative (How?)

Note $\Delta(h,k) - \Delta(h,0) = \Delta(h,k)$

$$\therefore \quad \Delta(h,k) = k\left[\partial_y f(a+h, b+tk) - \partial_y f(a, b+tk)\right] \quad — \text{(3)}$$

Rmk: The number $t \in (0,1)$ changes according to h, k. Different h,k yield different t.

Apply Mean value thm,

$$\partial_y f(a+h, b+tk) - \partial_y f(a, b+tk)$$

$$= h \cdot \partial_x \partial_y f(a+sh, b+tk) \quad \text{for some } s \in (0,1)$$

$\therefore$ For any (h,k), there are some $s,t \in (0,1)$ (depending on (h,k)) so that

$$\Delta(h,k) = hk\left[\partial_x \partial_y f(a+sh, b+tk)\right]$$

Since $\partial_x \partial_y f$ is continuous at (a,b),

$$\lim_{(h,k)\to(0,0)} \frac{\Delta(h,k)}{hk} = \frac{\partial^2 f}{\partial x \partial y}(a,b)$$

Recall that in the observation above,

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) = \lim_{k\to 0} \lim_{h\to 0} \frac{1}{hk} \Delta(h,k)$$

Note that

$\lim_{h\to 0} \frac{1}{hk} \Delta(h,k)$ always exists and

equals $\frac{1}{k}\left[\frac{\partial f}{\partial x}(a, b+k) - \frac{\partial f}{\partial x}(a,b)\right]$

Now, we have

① $\lim_{(h,k)\to(0,0)} \frac{\Delta(h,k)}{hk}$,

② $\lim_{h\to 0} \frac{\Delta(h,k)}{hk}$ exists $\forall k$

① & ② $\Rightarrow$ $\lim_{k\to 0} \lim_{h\to 0} \frac{\Delta(h,k)}{hk}$ exists

and equals $\lim_{(h,k)\to(0,0)} \frac{\Delta(h,k)}{hk}$

Rmk: The existence of $\lim_{k\to 0} \lim_{h\to 0} \frac{\Delta(h,k)}{hk}$

depends on ① & ②.